Visibility Graph Theory for Polygons

Subir Kumar Ghosh
School of Technology & Computer Science
Tata Institute of Fundamental Research
Mumbai 400005, India
ghosh@tifr.res.in
Overview

- Background
- Visibility Graph Recognition, Characterization, and Reconstruction
- Graph Theoretic Problems on Visibility Graphs
- Counting Visibility Graphs
- Conclusion


Let $P$ be a polygon with or without holes.

Two points $u$ and $w$ of $P$ are said to be visible if the line segment $uv$ is lies totally inside $P$.

Construct a graph $G$ from $P$ such that every vertex of $P$ is represented as a node in $G$ and two nodes are connected in $G$ iff their corresponding vertices in $P$ are visible from each other.

The graph $G$ is called the visibility graph of $P$. 
Algorithms for computing visibility graphs

1. T. Lozano-Perez and M. A. Wesley, *An algorithm for planning collision-free paths among polyhedral obstacles*, Communication of ACM, 22 (1979), 560-570. Running time: \( O(n^3) \).


4. E. Welzl, *Constructing the visibility graph for n line segments in O(n^2) time*, Information Processing Letters, 20 (1985), 167-171. Running time: \( O(n^2) \).


9. S. Kapoor and S. N. Maheshwari, *Efficiently Constructing the Visibility Graph of a Simple Polygon with Obstacles*, SIAM Journal on Computing, 30(2000), 847-871. Running time: $O(h \log n + T + E)$, where $T$ is the time for triangulation and $h$ is the number of holes.
Recognizing visibility graphs

Visibility graph recognition problem asks to determine if there is a polygon $P$ whose visibility graph is the given graph $G$.

Open Problem 1: Given a graph $G$ in adjacency matrix form, determine whether $G$ is the visibility graph of a simple polygon.

Open Problem 2: Is the problem of recognizing visibility graph in NP?

The visibility graph recognition problem has been solved for some special classes of polygons called spiral polygons and tower polygons.
A simple polygon is said to be *spiral* if its boundary consists of one chain of reflex vertices and one chain of convex vertices.

A vertex is *simplicial* if its neighborhood is a clique. There always exists two simplicial vertices (here, vertices $v_1$ and $v_6$) in the visibility graph of a spiral polygon.

Once a simplicial vertex is eliminated, another vertex in the remaining graph becomes simplicial. Here, the sequence of simplicial vertices is \((v_1, v_2, v_{11}, v_3, v_{10}, v_4, v_9, v_8, v_3, v_7, v_6)\).

Sequence of simplicial vertices forms a perfect vertex elimination scheme.

A graph is a *chordal* (or *triangulated*) if and only if it has a perfect vertex elimination scheme. So, visibility graphs of spiral polygons are chordal.
Three non-adjacent vertices are called an *asteroidal triple* if they cannot be ordered in such a way that every path from first vertex to the third vertex passes through a neighbor of the second vertex.

A graph is an *interval graph* if and only if the graph is a chordal graph containing no asteroidal triples.

Visibility graphs of spiral polygons do not contain any asteroidal triples and therefore, they are interval graphs.
A graph $G$ is the visibility graph of a spiral polygon if and only if (i) $G$ is an interval graph, and (ii) the paths corresponding to $L$ and $R$ formed by conductors in the ordering of maximal cliques in $G$ satisfy some geometric properties.

Visibility graphs of spiral polygons can be recognized in $O(n)$ time.

A tower polygon $F$ is a simple polygon formed by two reflex chains of vertices with only one boundary edge connecting two convex vertices.

Removing apex vertex $v_j$ and boundary edges of $F$ from the visibility graph $G$ of $F$ to form the cross-visible sub-graph $G'$ of $G$.

$G'$ is a bipartite permutation graph as it satisfies strong ordering.

A given graph $G$ is the visibility graph of a tower polygon if and only if its cross-visible sub-graph $G'$ is a bipartite permutation graph.
It takes $O(n)$ time to test whether $G'$ is a bipartite permutation graph.

A Hamiltonian cycle in $G$ which corresponds to the boundary of a tower polygon can be constructed in $O(n)$ time.

Visibility graphs of tower polygons can be recognized in $O(n)$ time.


Open Problem 3: Given a graph $G$ in adjacency matrix form along with a Hamiltonian cycle $C$ of $G$, determine whether $G$ is the visibility graph of a simple polygon $P$ whose boundary corresponds to the given Hamiltonian cycle $C$.

Observe that this problem is easier than the actual recognition problem as the edges of $G$ corresponding to boundary edges of $P$ have already been identified.

Assume that the vertices of $G$ are labeled with $v_1, v_2, \ldots, v_n$ and $C = (v_1, v_2, \ldots, v_n)$ is in counterclockwise order.
Ghosh’s necessary conditions

A cycle \( u_1, u_2, \ldots, u_k \) in \( G \) is said to be ordered if \( u_1, u_2, \ldots, u_k \) preserve their order in the Hamiltonian cycle. The Hamiltonian cycle is the longest ordered cycle in \( G \).

Necessary Condition 1. *In a visibility graph, every ordered cycle of \( k \geq 4 \) vertices has at least \( k - 3 \) chords.*

This condition is easily visualized: an ordered cycle corresponds to a sub-polygon and \( k \)-vertex sub-polygon must have a triangulation, which has \( k - 3 \) diagonals.
A vertex $v_a$ is a **blocking vertex** for an invisible pair $(v_i, v_j)$ if no two vertices $k \in chain(v_i, v_{a-1})$ and $m \in chain(v_{a+1}, v_j)$ are adjacent in $G$.

The vertex $v_a$ is called a blocking vertex because $v_a$ can be used to block the line of sight between $v_i$ and $v_j$ in the polygon.

**Necessary Condition 2.** *In a visibility graph, every invisible pair has at least one blocking vertex.*
Two invisible pairs \((v_i, v_j)\) and \((v_k, v_l)\) are called **separable** with respect to a vertex \(v_a\) if \(v_k\) and \(v_l\) are encountered before \(v_i\) and \(v_j\) when the Hamiltonian cycle is traversed from \(v_a\).

**Necessary Condition 3.** *In a visibility graph, two separable invisible pairs must have distinct blocking vertices.*

Ghosh conjectured in 1986 that the three necessary conditions are sufficient.

The vertex $v_2$ can block either $(v_1, v_8)$ or $(v_3, v_8)$. The vertex $v_4$ can block either $(v_3, v_8)$ or $(v_5, v_8)$.

**Necessary Condition 3'.** In a visibility graph, there is an assignment such that no blocking vertex $v_a$ is assigned to two or more minimal invisible pairs that are separable with respect to $v_a$.


Four vertices $v_1$, $v_2$, $v_5$, $v_8$ are reflex vertices in six vertices subpolygon $(v_1, v_2, v_4, v_5, v_8, v_9, v_1)$ since $v_1$, $v_2$, $v_5$, $v_8$ are assigned to invisible pairs $(v_2, v_9)$, $(v_1, v_4)$, $(v_4, v_8)$ and $(v_5, v_9)$ respectively.

Modified conjecture

**Necessary Condition 4.** Let $D$ be any ordered cycle of a visibility graph. For any assignment of blocking vertices to all minimal invisible pairs in the visibility graph, the total number of vertices of $D$ assigned to the minimal invisible pairs between the vertices of $D$ is at most $|D| - 3$.

**Ghosh’s Conjecture:** These four conditions (1, 2, 3′, and 4) are not only necessary, but also sufficient.

Testing necessary condition

Everett presented an $O(n^3)$ time algorithm for testing Necessary Condition 1 which was later improved by Ghosh to $O(n^2)$ time.

Ghosh also gave an $O(n^2)$ time algorithm for testing Necessary Condition 2.

Das et al. showed that Necessary Condition 3 can be tested in $O(n^4)$ time.

Open Problem 4: Design an algorithm for testing Necessary Condition 4 in polynomial time.

Another counter-example

This graph satisfies all four necessary conditions but it is not the visibility graph of any simple polygon.

The graph is given by Streinu as a counter-example to Ghosh’s new conjecture of sufficiency.

It is not clear whether either another necessary condition is required to circumvent this counter-example or there is a need to strengthen the existing necessary conditions.

We hope that the visibility graph recognition problem will be settled in the near future.

Characterizing visibility graphs

Visibility graph characterization problem asks to identify a set of properties satisfied by all visibility graphs.

Open Problem 5: Characterize visibility graphs of simple polygons.

Some results on this characterization problem are:

(i) ElGindy established that every maximal outerplanar graph is the visibility graph of a simple polygon.
(ii) Everett and Corneil characterized visibility graphs of spiral polygons as a subset of interval graphs.

(iii) Colley, Lubiw and Spinrad characterized visibility graphs of tower polygons as bipartite permutation graphs with an added Hamiltonian cycle.

(iv) Coullard and Lubiw have proved that every triconnected component of a visibility graph satisfies 3-clique ordering.

(v) Abello et al. characterized visibility graphs of staircase polygons.


Is visibility graph a perfect graph?

Ghosh’s Necessary Condition 1 suggests that every ordered cycle in a visibility graph has a chord. Is it true that every unordered cycle in a visibility graph has also a chord?

In the above figure, vertices $v_2$, $v_8$, $v_4$, $v_{10}$ and $v_6$ form an odd cycle without a diagonal.

Therefore, the chromatic number is not equal to the maximum cardinality clique in the visibility graph. Hence the visibility graph of a simple polygon is not a perfect graph (as well as chordal graph).
Is visibility graph a circle graph?

An undirected graph $G$ is called *circle graph* if there exists a set of chords $C$ on a circle and one-to-one correspondence between vertices of $G$ and chords of $C$ such that two distinct vertices are adjacent in $G$ if and only if their corresponding chords intersect.

A graph $G$ is not a circle graph if $G$ contains a wheel. Vertex $v_1$ and the cycle $v_2$, $v_6$, $v_5$, $v_4$ and $v_8$ have formed a wheel.
Ghosh tried to characterize visibility graphs in terms of perfect graphs, circle graphs or chordal graphs.

Coullard and Lubiw used clique ordering properties in their attempt to characterize visibility graphs.


Everett and Corneil attempted to characterize visibility graphs using forbidden induced sub-graphs.

Abello and Kumar attempted to characterize visibility graphs using Euclidean shortest paths.


Visibility graphs of simple polygons do not belong to the union of perfect graphs and circle graphs.

We feel that visibility graphs of simple polygons in general form a new class of graphs. However, the possibility of characterizing visibility graphs in terms of any known class of graphs cannot be totally ruled out.


Visibility graph reconstruction

The problem of actually drawing a simple polygon $P$ whose visibility graph is the given graph $G$, is called the visibility graph reconstruction problem for polygons.

Open Problem 6: Given a visibility graph $G$, draw a simple polygon whose visibility graph is $G$.

(i) Everett has shown that visibility graph reconstruction is in $PSPACE$. This is the only upper bound known on the complexity of the problem.

(ii) Relationship between visibility graphs and oriented matroids has been studied by Abello and Kumar.


(iii) Lin and Skiena studied the equivalent order types.

(iv) Streinu and O’Rourke et al. studied pseudo-line arrangements.


(v) Reconstruction problem for the visibility graphs of spiral polygons has been solved by Everett and Corneil.

(vi) Same problem for tower polygons has been solved by Choi, Shin and Chwa.


Reconstruction problem with added information has also been studied by several researchers.


A Hamiltonian cycle is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex.

The Hamiltonian cycle problem is to determine whether a Hamiltonian cycle exists in a given graph $G$.

Open Problem 7: Given the visibility graph $G$ of a simple polygon $P$, determine the Hamiltonian cycle in $G$ that corresponds to the boundary of $P$. 
The above graph $G$ is the visibility graph if the boundary of $P$ corresponds to the Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{10})$.

On the other hand, $G$ is not the visibility graph of any polygon $P$ as the boundary of $P$ corresponds to the Hamiltonian cycle $C = (v_1, v_5, v_9, v_3, v_4, v_6, v_8, v_{10}, v_2, v_7)$.

So, $G$ may be a valid visibility graph for a Hamiltonian cycle but $G$ may not be a valid visibility graph for another Hamiltonian cycle in $G$. 
Minimum dominating set in visibility graphs

A dominating set for a graph $G = (V, E)$ is a subset $D$ of $V$ such that every vertex not in $D$ is joined to at least one member of $D$ by some edge.

The minimum dominating set problem in visibility graphs corresponds to the art gallery problem in polygons which has been shown to be NP-hard.


Following the approximation algorithm for the art gallery problem for polygons given by Ghosh, a minimum dominating set of visibility graph can be computed with an approximation ratio of $O(\log n)$.

**Open Problem 8:** Design a constant factor approximation algorithm for computing minimum dominating set of visibility graphs.


Maximum hidden set in visibility graphs

An *independent set* is a set of vertices in a graph with no two of which are adjacent. Independent sets in visibility graphs are known as *hidden vertex sets*.

Shermer proved that the maximum hidden vertex set problem on visibility graphs is NP-hard.

**Open Problem 9:** Design an approximation algorithm for computing maximum hidden set of visibility graph.

However, the problem may not remain NP-hard if the Hamiltonian cycle corresponding to the boundary of the simple polygon is given as an input along with the visibility graph.

With this additional input, Ghosh, Shermer, Bhattacharya and Goswami showed that it is possible to compute in $O(n^3)$ time the maximum hidden vertex set in the visibility graph $G$ of a very special class of simple polygons called *convex fans*.

**Open Problem 10:** Given the visibility graph $G$ of a simple polygon $P$ along with the Hamiltonian cycle in $G$ corresponding to the boundary of $P$, determine the maximum hidden set of $G$.

Hidden vertex sets are also studied by (i) Eidenbenz, (ii) Ghosh, Maheshwari, Pal, Saluja and Veni Madhavan, and (iii) Lin and Skiena.


The **maximum clique** in $G$ is a complete subgraph of $G$ having the maximum number of vertices.

Vertices $v_1, v_3, v_5, v_6, v_7, v_8, v_9, v_{10}$, and $v_{11}$ have formed a maximum clique in $G$.

The maximum clique in $G$ corresponds to the largest empty convex polygon in $P$ that can be inscribed inside $P$ with the maximum number of vertices of $P$. 

---

**Maximum clique in visibility graph**

![Diagram of graph G and polygon P with vertices labeled]
The problem of computing the maximum clique in a visibility graph $G$ of a polygon $P$ with holes is known to be NP-hard.

However, the problem is not known to be NP-hard if the given graph $G$ is the visibility graph of a simple polygon.

Open Problem 11: Given the visibility graph $G$ of a simple polygon in an adjacency matrix form, compute a maximum clique in $G$ in polynomial time.

Given a set $S$ of $n$ points, the largest empty convex polygon in $S$ can be computed in $O(n^3)$ by the algorithm of Avis and Rappaport.

If the visibility graph $G$ and its corresponding polygon $P$ are given, the maximum clique in $G$ of $P$ can be computed in $O(n^3)$ time by the algorithm of Eidenbenz and Stamm.


Computing a maximum clique in a given visibility graph $G$ without any additional geometric input seems to be a hard problem.

If a simple polygon $P$ can be constructed from $G$ such that $G$ is its visibility graph, then a maximum clique in $G$ can be computed by the algorithm of Eidenbenz and Stamm.

However, there is no polynomial time algorithm known for constructing the corresponding simple polygon $P$ from a given visibility graph $G$.

Is it possible to compute a maximum clique in $G$ with additional information of $G$?
With additional information

- Assume that the Hamiltonian cycle $C$ in the visibility graph $G$ corresponding to the boundary of $P$ is given along with $G$.
- With this additional information, the maximum clique in $G$ can be computed in $O(n^2e)$ time by the algorithm of Ghosh et al., where $n$ and $e$ are number of vertices and edges in $G$ respectively.
- Note that there is no polynomial time algorithm known for locating the Hamiltonian cycle $C$ in $G$ that corresponds to the boundary of $P$.

Counting visibility graphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if and only if there is a bijection $f$ that maps vertices of $V_1$ to the vertices of $V_2$ such that an edge $(v, w) \in E_1$ if and only if the edge $(f(v), f(w)) \in E_2$.

It has been shown by Ghosh that the number of non-isomorphic visibility graphs of simple polygons of $n$ vertices is at least $2^{n-4}$.

On the other hand, Spinrad has pointed out that Warren’s theorem shows that the number of visibility graphs can be at most $2^{O(n \log n)}$.

Open Problem 12: Improve the upper and lower bounds on the number of non-isomorphic visibility graphs of simple polygons.

In this talk we have presented an overview of results on visibility graph theory for polygons and suggested several open problems. Hopefully, many more results will come which will enrich this fascinating area of geometric graph theory.